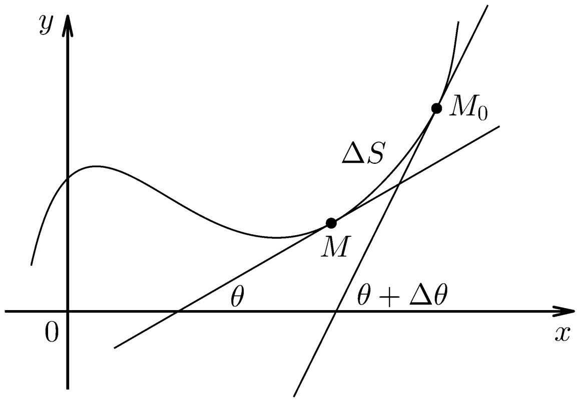
5 Zakrivljenost i torzija krive

Zakrivljenost ili krivina krive u ravni je veličina koja karakteriše stepen njenog odstupanja od prave u okolini neke tačke M. (Odstupanje čega?)



Pravac krive u tački M se može okarakterisati uglom θ koji gradi tangenta na krivu u tački M s osom $0x$ (vidi sliku). Brzina mjenjanja ugla θ pri ravnomjernom kretanju tačke M po krivoj naziva se krivina krive u tački M.

Torzija krive je brzina obrtanja oskulatorne ravni krive u tački A ako se tačka A kreće jedako (ravnomjerno) po krivoj brzinom jednakom jedinici. Na osnovu ove definicije postavlja se pitanje šta znaći ako je torzija uvijek jednaka nuli.

Date definicije krivine i torzije su opisne definicije.

Krivinu krive ćemo označiti sa K a poluprečnik krivine sa $R = \frac{1}{K}$. Torziju ćemo označavati sa $\frac{1}{T}$ (ili sa $-\tau$) a poluprečnik torzije sa $|T|$.

Poluprečnik krivine R i krivina K su određeni relacijama

$$\frac{1}{K^2} = R^2 = \frac{|\dot{\vec{r}}^2|^3}{|\dot{\vec{r}} \times \ddot{\vec{r}}|^2} \quad \text{tj. } R = \frac{|\vec{t}|^3}{|\vec{b}|}$$

Vidimo da vrijedi i

$$K = \frac{|\dot{\vec{r}} \times \ddot{\vec{r}}|}{|\dot{\vec{r}}|^3}.$$

Poluprečnik torzije, $\pm T$, je dat formulom:

$$T = -\frac{[\dot{\vec{r}} \times \ddot{\vec{r}}]^2}{\dot{\vec{r}}[\ddot{\vec{r}} \times \dddot{\vec{r}}]} = \frac{[\dot{\vec{r}} \times \ddot{\vec{r}}]^2}{\dddot{\vec{r}}\,[\dot{\vec{r}} \times \ddot{\vec{r}}]} = \frac{|\vec{b}|^2}{\dddot{\vec{r}} \cdot \vec{b}}$$

Torziju ćemo označiti sa $\frac{1}{T}$ (a nekad i sa $-\tau$). Ako je u jednačini krive parametar t jednak dužini luka s, tada je

$$K = \frac{1}{R} = \left|\frac{\mathrm{d}^2\vec{r}}{\mathrm{d}s^2}\right| = |\vec{r}''|$$

$$-\tau = \frac{1}{T} = \pm\left|\frac{\mathrm{d}\vec{b}_0}{\mathrm{d}s}\right|.$$

67. Odrediti jedinične vektore tangente, glavne normale i binormale krive

$$x = e^t cost,\ y = e^t sint,\ z = e^t.$$

Zatim naći krivinu i torziju date krive.

68. Pokazati da su kod krive

$$x = \mathrm{ch}\, z \qquad y = \mathrm{sh} z$$

radijus krive i torzije (R i T) jednaki.

69. Naći poluprečnik torzije $|T|$ za krivu

$$\vec{r} = \cos t\,\vec{i} + \sin t\,\vec{j} + \mathrm{sh} t\,\vec{k}.$$

70. Naći radijus krivine i krivinu krive

$$C : \begin{cases} x = \sin z - z\cos z \\ y = \cos z + z\sin z \end{cases}$$

u proizvoljnoj tački.

71. Napisati jednačinu skupa tačaka u kojima tangente zavojnice $\vec{r} = (a\cos t, a\sin t, bt)$ prodiru ravan $z = 0$. Odrediti zakrivljenost dobijene krive.

72. Izračunati torziju krive $\vec{r} = a(1 - \cos t, \sin 2t, 2\cos t)$ u proizvoljnoj tački. Odrediti jednačinu ravni kojoj kriva pripada.

73. Data je kriva

$$L : y = \frac{1}{2m}x^2,\ z = \frac{1}{6m^2}x^3$$

gdje je m parametar nezavisan od x i y. Naći ortove prirodnog triedra, fleksiju i torziju krive L u tački za koju je $x = 2m$.

74. Data je kriva $\vec{r} = \left\{\frac{t}{\sqrt{2}}, \frac{t}{\sqrt{2}}, \ln(\sin t)\right\}$. Odrediti ortove prirodnog triedra date krive. Odrediti fleksiju u proizvoljnoj tački krive.

75. Napisati jednačinu krive $\vec{r} = (a\cos t, a\sin t, bt)$ izrazivši $\vec{r}$ kao funkciju argumenta s. Diferenciranjem po luku s naći jedinične vektore tangente, glavn normale i binormale krive u proizvoljnoj tački. Izračunati krivinu i torziju krive u proizvoljnoj tački.

Odrediti jedinične vektore tangente, glavne normale i binormale krive

$$x = e^t \cos t, \quad y = e^t \sin t, \quad z = e^t.$$

Zatim naći krivinu i torziju date krive.

R: Vektore tangente, binormale i normale krive određujemo formulama $\vec{t} = \dot{\vec{r}}$, $\vec{b} = \dot{\vec{r}} \times \ddot{\vec{r}}$, $\vec{n} = \vec{b} \times \vec{t}$.

$$\vec{r} = (e^t \cos t,\ e^t \sin t,\ e^t)$$

$$\dot{\vec{r}} = (e^t \cos t - e^t \sin t,\ e^t \sin t + e^t \cos t,\ e^t)$$

$$\ddot{\vec{r}} = (e^t \cos t - e^t \sin t - e^t \sin t - e^t \cos t,\ e^t \sin t + e^t \cos t + e^t \cos t - e^t \sin t,\ e^t)$$

$$= (-2e^t \sin t,\ 2e^t \cos t,\ e^t)$$

$$\ddot{\vec{r}} = e^t(-2\sin t - 2\cos t,\ 2\cos t - 2\sin t,\ 1)$$

$$\vec{t} = e^t(\cos t - \sin t,\ \sin t + \cos t,\ 1)$$

$$|\vec{t}|^2 = e^{2t}(\cos^2 t \cancel{- 2\cos t \sin t} + \sin^2 t + \sin^2 t \cancel{+ 2\sin t \cos t} + \cos^2 t + 1) =$$

$$= 3e^{2t} \Rightarrow |\vec{t}| = e^t\sqrt{3}$$

Jedinični vektor tangente je

$$\vec{t}_0 = \frac{\vec{t}}{|\vec{t}|} = \frac{1}{\sqrt{3}}(\cos t - \sin t,\ \sin t + \cos t,\ 1)$$ jedinični vektor tangente

$$\vec{b} = \dot{\vec{r}} \times \ddot{\vec{r}} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ e^t \cos t - e^t \sin t & e^t \sin t + e^t \cos t & e^t \\ -2e^t \sin t & 2e^t \cos t & e^t \end{vmatrix} =$$

$$= e^t \cdot e^t \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos t - \sin t & \sin t + \cos t & 1 \\ -2\sin t & 2\cos t & 1 \end{vmatrix} =$$

$$= e^{2t}(\sin t + \cos t - 2\cos t,\ -(\cos t - \sin t + 2\sin t),\ 2\cos^2 t \cancel{- 2\sin t \cos t} + 2\sin^2 t \cancel{+ 2\sin t \cos t})$$

$$\vec{b} = e^{2t}(\sin t - \cos t, -\sin t - \cos t, 2)$$

$$|\vec{b}|^2 = e^{4t}\cdot(\sin^2 t - \cancel{2\sin t\cos t} + \cos^2 t + \sin^2 t + \cancel{2\sin t\cos t} + \cos^2 t + 4)$$

$$|\vec{b}| = e^{2t}\sqrt{6}$$

$$\vec{b}_0 = \frac{\vec{b}}{|\vec{b}|} = \frac{1}{\sqrt{6}}(\sin t - \cos t, -(\sin t + \cos t), 2)$$ jedinični vektor binormale

$$\vec{n}_0 = \vec{b}_0 \times \vec{t}_0 = \frac{1}{\sqrt{6}}\cdot\frac{1}{\sqrt{3}}\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \sin t - \cos t & -\sin t - \cos t & 2 \\ \cos t - \sin t & \sin t + \cos t & 1 \end{vmatrix} =$$

$$= \frac{1}{\sqrt{18}}\left(-\sin t - \cos t + 2\sin t - 2\cos t, -(\sin t - \cos t - 2\cos t + 2\sin t), \sin^2 t - \cos^2 t + \cos^2 t - \sin^2 t\right) =$$

$$= \frac{1}{3\sqrt{2}}\left((-3)(\sin t + \cos t), (-3)\cdot(\sin t - \cos t), 0\right) =$$

$$\vec{n}_0 = \frac{1}{\sqrt{2}}\left((-1)(\sin t + \cos t), -\sin t + \cos t, 0\right)$$ jedinični vektor glavne normale

Krivinu krive možemo izračunati po formuli $K = \frac{1}{R}$ gde je R poluprečnik krivine $R = \frac{|\vec{t}|^3}{|\vec{b}|}$

$$R = \frac{(e^t\sqrt{3})^3}{e^{2t}\sqrt{6}} = \frac{e^{3t}3\sqrt{3}}{e^{2t}\sqrt{6}} = 3e^t\sqrt{\frac{1}{2}} \Rightarrow K = \frac{\sqrt{2}}{3e^t}$$ tražena krivina krive

Torziju možemo izračunati po formuli $-\tau = \frac{1}{T} = \frac{\dddot{\vec{r}}\cdot\vec{b}}{|\vec{b}|^2}$

$$\dddot{\vec{r}} = e^t(-2\sin t - 2\cos t, \overbrace{2\cos t - 2\sin t}^{2(\cos t - \sin t)}, 1)$$

$$\vec{b} = e^{2t}(\sin t - \cos t, \underbrace{-\sin t - \cos t}_{(-1)(\cos t + \sin t)}, 2)$$

$$\dddot{\vec{r}}\cdot\vec{b} = e^{3t}\left((-2)(\sin^2 t - \cos^2 t) - 2(\cos^2 t - \sin^2 t) + 2\right) = 2e^{3t}$$

$$-\tau = \frac{2e^{3t}}{e^{4t}\sqrt{6}} = \sqrt{\frac{4}{6}}\cdot\frac{1}{e^t} = \frac{\sqrt{2}}{e^t\sqrt{3}}$$ tražena torzija krive

(#) Pokazati da su kod krive

$$x = \operatorname{ch} z \qquad y = \operatorname{sh} z$$

radijus krive i torzije (R i T) jednaki.

Rj. Ako uvedemo smjenu $z = t$ datu krivu možemo napisati u obliku

$$C \searrow \vec{r} : \begin{cases} x = \operatorname{ch} t \\ y = \operatorname{sh} t \\ z = t \end{cases}$$

Tada je

$$\dot{\vec{r}} = (\operatorname{sh} t, \operatorname{ch} t, 1)$$
$$\ddot{\vec{r}} = (\operatorname{ch} t, \operatorname{sh} t, 0)$$
$$\dddot{\vec{r}} = (\operatorname{sh} t, \operatorname{ch} t, 0)$$

Neka je $\dot{\vec{r}} = \dot{\vec{r}}(\dot{x}, \dot{y}, \dot{z})$

$$\ddot{\vec{r}} = \ddot{\vec{r}}(\ddot{x}, \ddot{y}, \ddot{z})$$
$$\dddot{\vec{r}} = \dddot{\vec{r}}(\dddot{x}, \dddot{y}, \dddot{z})$$

Tada je

$$R^2 = \frac{\left(\dot{\vec{r}}^{\,2}\right)^3}{[\dot{\vec{r}} \times \ddot{\vec{r}}]^2}, \quad a \quad T = \frac{-[\dot{\vec{r}} \times \ddot{\vec{r}}]^2}{\dot{\vec{r}}(\ddot{\vec{r}} \times \dddot{\vec{r}})}$$

$$\dot{\vec{r}}^{\,2} = \dot{\vec{r}} \cdot \dot{\vec{r}} = \operatorname{sh}^2 t + \operatorname{ch}^2 t + 1 = 2\operatorname{ch}^2 t \qquad \boxed{\operatorname{sh}^2 t + 1 = \operatorname{ch}^2 t}$$

$$\dot{\vec{r}} \times \ddot{\vec{r}} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \operatorname{sh} t & \operatorname{ch} t & 1 \\ \operatorname{ch} t & \operatorname{sh} t & 0 \end{vmatrix} = (-\operatorname{sh} t, \operatorname{ch} t, \operatorname{sh}^2 t - \operatorname{ch}^2 t)$$
$$= (-\operatorname{sh} t, \operatorname{ch} t, -1)$$

$$(\dot{\vec{r}} \times \ddot{\vec{r}})^2 = \operatorname{sh}^2 t + \operatorname{ch}^2 t + 1 = 2\operatorname{ch}^2 t$$

$$R^2 = \frac{(2\operatorname{ch}^2 t)^3}{2\operatorname{ch}^2 t} = (2\operatorname{ch}^2 t)^2 \quad \Rightarrow \quad R = |2\operatorname{ch}^2 t| = 2\operatorname{ch}^2 t$$

$$\dot{\vec{r}}(\ddot{\vec{r}} \times \dddot{\vec{r}}) = \begin{vmatrix} \operatorname{sh} t & \operatorname{ch} t & 1 \\ \operatorname{ch} t & \operatorname{sh} t & 0 \\ \operatorname{sh} t & \operatorname{ch} t & 0 \end{vmatrix} = -1 \underbrace{(\operatorname{ch}^2 t - \operatorname{sh}^2 t)}_{-1} = +1$$

$$T = +\frac{2\operatorname{ch}^2 t}{+1} = 2\operatorname{ch}^2 t$$

$$R = |T| = 2\operatorname{ch}^2 t$$

traženo rješenje

(#) Naći poluprečnik torzije $|T|$ za krivu

$$\vec{r} = \cos t\,\vec{i} + \sin t\,\vec{j} + \operatorname{sh} t\,\vec{k}.$$

Rj.

Poluprečnik torzije $|T|$ možemo naći po formuli

$$T = -\frac{[\dot{\vec{r}} \times \ddot{\vec{r}}]^2}{\dot{\vec{r}}\,[\ddot{\vec{r}} \times \dddot{\vec{r}}]}$$

$$\dot{\vec{r}} = (-\sin t, \cos t, \operatorname{ch} t)$$

$$\ddot{\vec{r}} = (-\cos t, -\sin t, \operatorname{sh} t)$$

$$\dddot{\vec{r}} = (\sin t, -\cos t, \operatorname{ch} t)$$

$$\dot{\vec{r}} \times \ddot{\vec{r}} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -\sin t & \cos t & \operatorname{ch} t \\ -\cos t & -\sin t & \operatorname{sh} t \end{vmatrix} = (\cos t \operatorname{sh} t + \sin t \operatorname{ch} t,\ \sin t \operatorname{sh} t - \cos t \operatorname{ch} t,\ 1)$$

pa je

$$[\dot{\vec{r}} \times \ddot{\vec{r}}]^2 = \cos^2 t \operatorname{sh}^2 t + \underline{2 \sin t \cancel{\cos t \operatorname{sh} t} \operatorname{ch} t} + \sin^2 t \operatorname{ch}^2 t$$
$$+ \sin^2 t \operatorname{sh}^2 t - \underline{2 \sin t \cancel{\cos t \operatorname{sh} t} \operatorname{ch} t} + \cos^2 t \operatorname{ch}^2 t + 1 =$$

$$= \operatorname{sh}^2 t + \operatorname{ch}^2 t + 1 = \operatorname{ch}^2 t + \left(\frac{e^t - e^{-t}}{2}\right)^2 + 1 = \operatorname{ch}^2 t + \frac{e^{2t} - 2 + e^{-2t}}{4} + 1$$
$$= \operatorname{ch}^2 t + \frac{e^{2t} + 2 + e^{-2t}}{4} = 2 \operatorname{ch}^2 t$$

$$\dot{\vec{r}}\,[\ddot{\vec{r}} \times \dddot{\vec{r}}] = \begin{vmatrix} -\sin t & \cos t & \operatorname{ch} t \\ -\cos t & -\sin t & \operatorname{sh} t \\ \sin t & -\cos t & \operatorname{ch} t \end{vmatrix} \overset{I_v + III_v}{=\!=} \begin{vmatrix} 0 & 0 & 2\operatorname{ch} t \\ -\cos t & -\sin t & \operatorname{sh} t \\ \sin t & -\cos t & \operatorname{ch} t \end{vmatrix} = 2 \operatorname{ch} t$$

$$|T| = \left| -\frac{2 \operatorname{ch}^2 t}{2 \operatorname{ch} t} \right| = \operatorname{ch} t = \frac{e^t + e^{-t}}{2}$$

traženi poluprečnik torzije

(#) Naći radijus krivine i krivinu krive

$$C: \begin{cases} x = \sin z - z\cos z \\ y = \cos z + z\sin z \end{cases}$$

u proizvoljnoj tački.

Rj. Kao parametar stavimo $z=t$. Tada

$$\vec{r} = (\sin t - t\cos t,\ \cos t + t\sin t,\ t)$$

Krivina krive K je data izrazom $K = \dfrac{|\dot{\vec{r}} \times \ddot{\vec{r}}|}{|\dot{\vec{r}}|^3}$,

a poluprečnik krivine je $R = \dfrac{1}{K}$.

$$\dot{\vec{r}} = (\cos t - \cos t + t\sin t,\ -\sin t + \sin t + t\cos t,\ 1)$$
$$= (t\sin t,\ t\cos t,\ 1)$$

$$\ddot{\vec{r}} = (\sin t + t\cos t,\ \cos t - t\sin t,\ 0)$$

$$\dot{\vec{r}} \times \ddot{\vec{r}} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ t\sin t & t\cos t & 1 \\ \sin t + t\cos t & \cos t - t\sin t & 0 \end{vmatrix} = (t\sin t - \cos t,\ \sin t + t\cos t,\ \underline{t\sin t\cos t} - t^2\sin^2 t \underline{- t\sin t\cos t} - t^2\cos^2 t)$$

$$= (t\sin t - \cos t,\ \sin t + t\cos t,\ -t^2)$$

$$|\dot{\vec{r}} \times \ddot{\vec{r}}|^2 = (\dot{\vec{r}} \times \ddot{\vec{r}})^2 = t^2\sin^2 t \cancel{- 2t\sin t\cos t} + \underbrace{\cos^2 t + \sin^2 t}_{=1} \cancel{+ 2t\sin t\cos t} + t^2\cos^2 t$$
$$+ t^4 = 1 + t^2 + t^4$$

$$|\dot{\vec{r}}|^2 = t^2\sin^2 t + t^2\cos^2 t + 1 = t^2 + 1$$

$$|\dot{\vec{r}}| = \sqrt{t^2+1}$$

$$K = \frac{\sqrt{1+t^2+t^4}}{\sqrt{(t^2+1)^3}} \quad ; \quad R = \frac{\sqrt{(t^2+1)^3}}{\sqrt{1+t^2+t^4}}$$

Napisati jednačinu skupa tačaka u kojima tangente zavojnice $\vec{r}=(a\cos t,\ a\sin t,\ bt)$ prodiru ravan $z=0$. Odrediti zakrivljenost dobijene krive.

Rj: Pronađimo prvo jednačinu tangente na zavojnicu u proizvoljnoj tački $M(t)$.

$$\frac{d\vec{r}}{dt}=(-a\sin t,\ a\cos t,\ b)$$

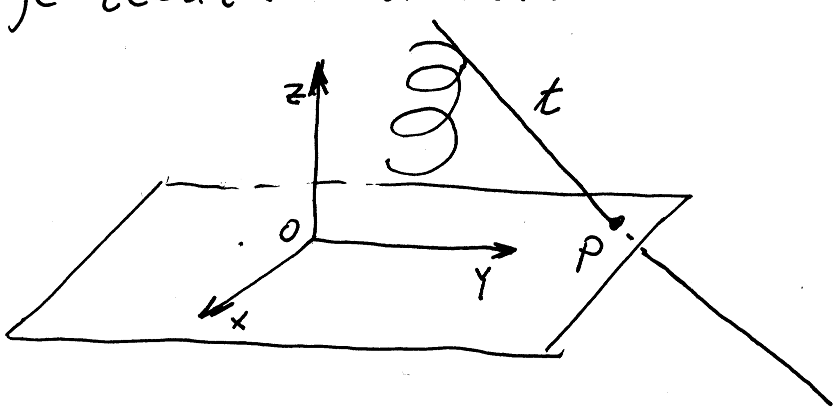
Ako uzmemo tačku $M(a\cos t,\ a\sin t,\ bt)$ imamo sljedeću jednačinu tangente

$$t:\quad \frac{x-a\cos t}{-a\sin t}=\frac{y-a\sin t}{a\cos t}=\frac{z-bt}{b}$$

U parametarskom obliku, gdje je u parametar, imamo

$$t:\ \begin{cases} x=a\cos t-au\sin t\\ y=a\sin t+au\cos t\\ z=bt+ub\\ \quad u\in\mathbb{R}\end{cases}$$

u je tekuća koordinata za tačke tangente



Prodor P tangente $\vec{t}$ sa ravni $z=0$ dobija se za $bt+ub=0$, tj. za $u=-t$.

Dakle koordinate tačke prodora P su

$$(a\cos t+at\sin t,\ a\sin t-at\cos t,\ 0)$$

Za različitu vrijednost parametra t drugu tačku na zavojnici, drugačiju tangentu na zavojnicu a samim time i drugačiju tačku prodora.

Skup tačaka u kojima tangente zavojnice $\vec{r}$ prodiru ravan $z=0$ formiraju sljedeću krivu

$$\vec{r}^{\,*}: \begin{cases} x = a\cos t + at\sin t \\ y = a\sin t - at\cos t \\ z = 0 \\ -\infty < t < +\infty \end{cases}$$

Sad trebamo odrediti zakrivljenost dobijene krive

$$\vec{t}^{\,*} = \dot{\vec{r}}^{\,*} = (-\cancel{a\sin t} + \cancel{a\sin t} + at\cos t,\ \cancel{a\cos t} - \cancel{a\cos t} + at\sin t,\ 0)$$

Krivina K je određena relacijom $K = \frac{1}{R}$ gdje je R poluprečnik krivine određen relacijom $R = \frac{|\vec{t}|^3}{|\vec{b}|}$.

$$\vec{b}^{\,*} = \dot{\vec{r}}^{\,*} \times \ddot{\vec{r}}^{\,*} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ at\cos t & at\sin t & 0 \\ a\cos t - at\sin t & a\sin t + at\cos t & 0 \end{vmatrix} \stackrel{(**)}{=}$$

$$\ddot{\vec{r}}^{\,*} = (a\cos t - at\sin t,\ a\sin t + at\cos t,\ 0)$$

$$\stackrel{(**)}{=} a^2 \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ t\cos t & t\sin t & 0 \\ \cos t - t\sin t & \sin t + t\cos t & 0 \end{vmatrix} = a^2(0, 0, \underline{t\sin t\cos t} + t^2\cos^2 t \underline{- t\sin t\cos t} + t^2\sin^2 t)$$

$$= a^2(0, 0, t^2)$$

$$= a^2(0, 0, t^2)$$

$$|\vec{t}^{\,*}| = \sqrt{a^2t^2\cos^2 t + a^2t^2\sin^2 t + 0} = at$$

$$|\vec{t}^{\,*}|^3 = a^3t^3$$

$$|\vec{b}^{\,*}| = a^2\sqrt{0+0+t^4} = a^2t^2$$

$$R = \frac{a^3t^3}{a^2t^2} = at \qquad K = \frac{1}{at}$$ tražena zakrivljenost date krive

(#) Izračunati torziju krive $\vec{r} = a(1-\cos t, \sin 2t, 2\cos t)$ u proizvoljnoj tački. Odrediti jednačinu ravni kojoj kriva pripada.

Rj: Torziju krive možemo izračunati po formuli

$$\frac{1}{T} = -\frac{\dot{\vec{r}}\,[\ddot{\vec{r}} \times \dddot{\vec{r}}]}{|\dot{\vec{r}} \times \ddot{\vec{r}}|^2}$$

Kako je

$$\dot{\vec{r}}\,[\ddot{\vec{r}} \times \dddot{\vec{r}}] = a^3 \begin{vmatrix} \sin t & 2\cos 2t & -2\sin t \\ \cos t & -4\sin 2t & -2\cos t \\ -\sin t & -8\cos 2t & 2\sin t \end{vmatrix} \overset{III_k + I_k \cdot 2}{=\!=} 0$$

To je torzija $\frac{1}{T} = 0$

Ako je torzija $\frac{1}{T} = 0$ u svakoj tački krive, onda kriva leži u ravni. Ta ravan u ovom slučaju ima jednačinu $Ax + By + Cz + D = 0$.

U našem slučaju

$$A(a - a\cos t) + Ba\sin 2t + C2a\cos t + D = 0 \qquad /:a$$

$$(A+D) + (2C - A)\cos t + B\sin 2t = 0 \qquad \forall t$$

$$A + D = 0, \quad 2C - A = 0, \quad B = 0 \quad \text{tj.}$$

$$D = -A, \quad C = \frac{A}{2}, \quad B = 0$$

$$Ax + \frac{A}{2}z - A = 0 \qquad /:A$$

$$x + \frac{1}{2}z - 1 = 0$$

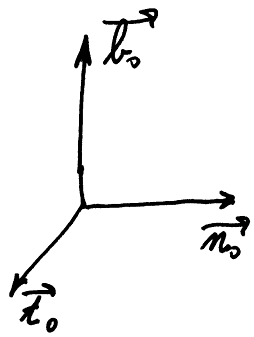
(#) Data je kriva

$$L: y=\frac{1}{2m}x^2, \quad z=\frac{1}{6m^2}x^3$$

gdje je m parametar nezavisan od x i y. Nađi ortove prirodnog triedra, fleksiju i torziju krive L u tački za koju je $x=2m$.

Rj: Jednačina krive L u vektorskom obliku glasi

$$\vec{r}=\left\{t,\ \frac{1}{2m}t^2,\ \frac{1}{6m^2}t^3\right\}$$



prirodni triedar

Znamo da vrijedi

$$\vec{t}=\dot{\vec{r}}$$

$$\vec{b}=\dot{\vec{r}}\times\ddot{\vec{r}}$$

$$\vec{n}=\vec{b}\times\vec{t}$$

određeni relacijama

Ortovi prirodnog triedra su $\vec{t}_0=\dfrac{\vec{t}}{|\vec{t}|},\quad \vec{b}_0=\dfrac{\vec{b}}{|\vec{b}|},\quad \vec{n}_0=\dfrac{\vec{n}}{|\vec{n}|}$

$$\dot{\vec{r}}=\left\{1,\ \frac{1}{m}t,\ \frac{1}{2m^2}t^2\right\}$$

$$\ddot{\vec{r}}=\left\{0,\ \frac{1}{m},\ \frac{t}{m^2}\right\} \qquad \dddot{\vec{r}}=\left\{0,\ 0,\ \frac{1}{m^2}\right\}$$

Za $x=2m$ je

$$\dot{\vec{r}}=\{1,2,2\},$$

$$\ddot{\vec{r}}=\left\{0,\ \frac{1}{m},\ \frac{2}{m}\right\}$$

$$\dddot{\vec{r}}=\left\{0,\ 0,\ \frac{1}{m^2}\right\}$$

pa je $\vec{t} = (1,2,2)$, $|\vec{t}| = \sqrt{9} = 3$

$$\vec{b} = \dot{\vec{r}} \times \ddot{\vec{r}} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 2 \\ 0 & \frac{1}{m} & \frac{2}{m} \end{vmatrix} = \left(\frac{2}{m}, \frac{-2}{m}, \frac{1}{m}\right)$$

$$|\vec{b}| = \sqrt{\frac{9}{m^2}} = \frac{3}{|m|}$$

$$\vec{n} = \vec{b} \times \vec{t} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{2}{m} & -\frac{2}{m} & \frac{1}{m} \\ 1 & 2 & 2 \end{vmatrix} = \left(-\frac{6}{m}, -\frac{3}{m}, \frac{6}{m}\right)$$

$$|\vec{n}| = \sqrt{\frac{81}{m^2}} = \frac{9}{|m|}$$

Prema tome

$$\vec{t} = \frac{1}{3}(\vec{i} + 2\vec{j} + 2\vec{k}), \quad \vec{b} = \frac{1}{3}(2\vec{i} - 2\vec{j} + \vec{k}), \quad \vec{n} = \frac{1}{3}(-2\vec{i} - \vec{j} + 2\vec{k})$$

Fleksiju i torziju možemo izračunati po obrascima

$$K = \frac{|\dot{\vec{r}} \times \ddot{\vec{r}}|}{|\dot{\vec{r}}|^3} \qquad \tau = -\frac{1}{T} = \frac{\dot{\vec{r}}(\ddot{\vec{r}} \times \dddot{\vec{r}})}{|\dot{\vec{r}} \times \ddot{\vec{r}}|^2}$$

$$|\dot{\vec{r}} \times \ddot{\vec{r}}| = \frac{3}{|m|}, \quad |\dot{\vec{r}}| = 3, \quad |\dot{\vec{r}}|^3 = 27 \quad \Rightarrow \quad K = \frac{1}{9|m|}.$$

$$\dot{\vec{r}}(\ddot{\vec{r}} \times \dddot{\vec{r}}) = \begin{vmatrix} 1 & 2 & 2 \\ 0 & \frac{1}{m} & \frac{2}{m} \\ 0 & 0 & \frac{1}{m^2} \end{vmatrix} = 1 \cdot \frac{1}{m} \cdot \frac{1}{m^2} = \frac{1}{m^3}, \quad |\dot{\vec{r}} \times \ddot{\vec{r}}|^2 = \frac{9}{m^2}$$

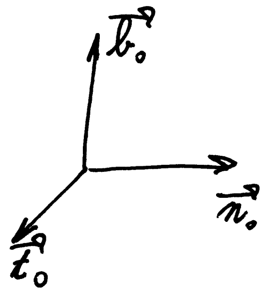
$$\tau = \frac{1}{9m}$$

Data je kriva

$$\vec{r} = \left\{ \frac{t}{\sqrt{2}}, \frac{t}{\sqrt{2}}, \ln \sin t \right\}$$

Odrediti ortove prirodnog triedra date krive.
Odrediti fleksiju u proizvoljnoj tački krive.

Rj.



prirodni triedar

Znamo da vrijedi

$$\vec{t} = \dot{\vec{r}}$$

$$\vec{b} = \dot{\vec{r}} \times \ddot{\vec{r}}$$

$$\vec{n} = \vec{b} \times \vec{t}$$

<u>Ortovi prirodnog triedra su određeni relacijama</u>

$$\vec{t}_0 = \frac{\vec{t}}{|\vec{t}|}, \quad \vec{b}_0 = \frac{\vec{b}}{|\vec{b}|}, \quad \vec{n}_0 = \frac{\vec{n}}{|\vec{n}|}$$

$$\dot{\vec{r}} = \left\{ \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \overset{\operatorname{ctg} t}{\frac{\cos t}{\sin t}} \right\} \Rightarrow \vec{t} = \left\{ \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{\cos t}{\sin t} \right\}$$

$$|\vec{t}| = \sqrt{\frac{1}{2} + \frac{1}{2} + \frac{\cos^2 t}{\sin^2 t}} = \frac{1}{|\sin t|}$$

$$\ddot{\vec{r}} = \left\{ 0, 0, -\frac{1}{\sin^2 t} \right\}$$

$$\dddot{\vec{r}} = \left\{ 0, 0, \frac{2\cos t}{\sin^3 t} \right\}$$

$$\vec{b} = \dot{\vec{r}} \times \ddot{\vec{r}} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \operatorname{ctg} t \\ 0 & 0 & -\frac{1}{\sin^2 t} \end{vmatrix} =$$

$$= \left(-\frac{1}{\sqrt{2}\sin^2 t}, +\frac{1}{\sqrt{2}\sin^2 t}, 0 \right)$$

$$|\vec{b}| = \sqrt{\frac{1}{2\sin^4 t} + \frac{1}{2\sin^4 t}}$$

$$= \frac{1}{\sin^2 t}$$

$$\vec{n} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -\frac{1}{\sqrt{2}\sin^2 t} & \frac{1}{\sqrt{2}\sin^2 t} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{\cos t}{\sin t} \end{vmatrix} = \left(\frac{1}{\sqrt{2}\sin^3 t}, -\frac{\cos t}{\sqrt{2}\sin^3 t}, -\frac{1}{2\sin^2 t} - \frac{1}{2\sin^2 t} \right)$$

$$= \left(\frac{1}{\sqrt{2}\sin^3 t}, -\frac{\cos t}{\sqrt{2}\sin^3 t}, -\frac{1}{\sin^2 t} \right)$$

$$|\vec{n}|^2 = \frac{1}{2\sin^6 t} + \frac{\cos^2 t}{2\sin^6 t} + \frac{1}{\sin^4 t} = \frac{1 + \cos^2 t + \sin^2 t}{2\sin^6 t}$$

$$= \frac{1}{\sin^6 t} \quad \Rightarrow \quad |n| = \frac{1}{|\sin t| \sin^2 t}$$

Prema tome

$$\vec{t}_0 = |\sin t| \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \operatorname{ctg} t \right)$$

$$\vec{b}_0 = \frac{1}{\sqrt{2}} (-1, 1, 0)$$

$$\vec{n}_0 = \frac{|\sin t|}{\sqrt{2}} (\operatorname{ctg} t, \operatorname{ctg} t, -\sqrt{2})$$

orbovi prirodnog triedra

Fleksiju možemo izračunati po formuli

$$K = \frac{|\dot{\vec{r}} \times \ddot{\vec{r}}|}{|\dot{\vec{r}}|^3} \qquad |\dot{\vec{r}} \times \ddot{\vec{r}}| = \frac{1}{\sin^2 t}$$

$$|\dot{\vec{r}}| = \frac{1}{|\sin t|}$$

$$K = \frac{\frac{1}{\sin^2 t}}{\frac{1}{\sin^2 t |\sin t|}} = |\sin t|$$

fleksija krive

(#) Napisati jednačinu krive $\vec{r}=(a\cos t, a\sin t, bt)$ izrazivši $\vec{r}$ kao funkciju argumenta s. Diferenciranjem po luku s naći jedinične vektore tangente, glavne normale i binormale krive u proizvoljnoj tački. Izračunati krivinu i torziju krive u proizvoljnoj tački.

Rj.

Znamo da je

$$ds=\sqrt{x'^2+y'^2+z'^2}\,dt$$

$$x'^2+y'^2+z'^2=a^2\sin^2 t+a^2\cos^2 t+b^2=a^2+b^2 \Rightarrow ds=\sqrt{a^2+b^2}\,dt$$

$$ds=\sqrt{a^2+b^2}\,dt \Rightarrow s=\sqrt{a^2+b^2}\,t \Rightarrow t=\frac{s}{\sqrt{a^2+b^2}}$$

$$\vec{r}=\left(a\cos\frac{s}{\sqrt{a^2+b^2}}, a\sin\frac{s}{\sqrt{a^2+b^2}}, \frac{bs}{\sqrt{a^2+b^2}}\right).$$

Vektor tangente sad računamo po formuli $\vec{t}=\frac{d\vec{r}}{ds}$

$$\frac{d\vec{r}}{ds}=\left(-a\cdot\frac{1}{\sqrt{a^2+b^2}}\sin\frac{s}{\sqrt{a^2+b^2}}, a\frac{1}{\sqrt{a^2+b^2}}\cos\frac{s}{\sqrt{a^2+b^2}}, \frac{b}{\sqrt{a^2+b^2}}\right)$$

$$=\frac{1}{\sqrt{a^2+b^2}}\left(-a\sin\frac{s}{\sqrt{a^2+b^2}}, a\cos\frac{s}{\sqrt{a^2+b^2}}, b\right).$$

Primjetimo da smo u stvari dobili jedinični vektor tangente.

Iz $\frac{d^2\vec{r}}{ds^2}=\frac{d\vec{t}}{ds}=\frac{1}{a^2+b^2}\left(-a\cos\frac{s}{\sqrt{a^2+b^2}}, -a\sin\frac{s}{\sqrt{a^2+b^2}}, 0\right)$

dobija se jedinični vektor glavne normale

$$\vec{n}=\frac{\frac{d^2\vec{r}}{ds^2}}{\left|\frac{d^2\vec{r}}{ds^2}\right|}=\left(-\cos\frac{s}{\sqrt{a^2+b^2}}, -\sin\frac{s}{\sqrt{a^2+b^2}}, 0\right)$$

Jedinični vektor binormale

$$\vec{b} = \vec{t} \times \vec{n} = \frac{1}{\sqrt{a^2+b^2}} \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -a\sin\frac{s}{a^2+b^2} & a\cos\frac{s}{a^2+b^2} & b \\ -\cos\frac{s}{\sqrt{a^2+b^2}} & -\sin\frac{s}{a^2+b^2} & 0 \end{vmatrix}$$

$$= \frac{1}{\sqrt{a^2+b^2}}\left(b\sin\frac{s}{\sqrt{a^2+b^2}}, -b\cos\frac{s}{\sqrt{a^2+b^2}}, a\right)$$

Kako je krivina $K = \left|\frac{d^2\vec{r}}{ds^2}\right|$ to je $K = \frac{a}{a^2+b^2}$

Kako je $\frac{d\vec{b}_0}{ds} = \frac{\vec{n}_0}{T}$ (gdje je $\frac{1}{T} = -\tau$ torzija)

to se torzija krive može odrediti iz

$$\frac{d\vec{b}}{ds} = -\tau\vec{n}$$ (u našem slučaju $\vec{b}$ i $\vec{n}$ su jedinični vektori)

$$\frac{d\vec{b}}{ds} = \frac{1}{a^2+b^2}\left(b\cos\frac{s}{\sqrt{a^2+b^2}}, b\sin\frac{s}{\sqrt{a^2+b^2}}, 0\right) = -\frac{b}{a^2+b^2}\vec{n}$$

$$\Rightarrow \quad \tau = \frac{b}{a^2+b^2}$$

tražena torzija krive